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DENSIFICATION BY PLASTIC DEFORMATION AROUND SPHERICAL INCLUSIONS

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Abstract—The densification of a metal powder compact containing hard spherical inclusions is modeled. The inclusions are rigid and isolated, while the surrounding matrix material deforms according to a homogeneous non-hardening compressible plasticity description. Solutions are obtained for the limiting cases of small and large inclusion volume fractions and approximate solutions are derived for intermediate inclusion concentrations. These analytic results are shown to be in good agreement with complete finite difference solutions. The presence of inclusions increases the work required to densify the matrix by less than 15%, however the density is shown to asymptotically approach a non-uniform distribution.

1. INTRODUCTION

A composite material, consisting of hard spherical particles in a metal matrix can be created by forming a mixture of the matrix material, in a powdered form, and the particulate inclusions. This powder compact can then be densified by the application of a combination of pressure and temperature. When the primary mechanism is diffusion, the process is known as sintering. In this analysis we consider the application of a large hydrostatic stress to the compact such that the densification mechanism is time-independent plastic deformation. At elevated temperatures, when the deformation is time dependent, the process is known as hot isostatic pressing or HIPing. Ashby (1990) discussed the various mechanisms of densification.

We consider the powdered matrix material to be a voided material which deforms according to the compressible plasticity model proposed by Gurson (1977). The Gurson model has been derived for relatively small levels of porosity. Ashby (1990) describes small porosity densification as stage 2, or final stage densification. Here, we consider a non-hardening material and further assume that elastic effects are negligible, so that the material can be modeled as rigid-perfectly plastic. The composite compact is then modeled as a single rigid spherical inclusion surrounded by a porous spherical shell described as a rigid-perfectly plastic Gurson material. Mear and Durban (1989) consider the similar problem of the spherically symmetric deformation of a Gurson material shell with a stress-free inner boundary.

We begin by analysing several uniform stress and deformation states that relate to limiting cases for the densification of particulate composites. Non-uniform analytic solutions are obtained for the spherical problem for an initial uniform porosity state as well as asymptotically as the porosity approaches zero. Finally, the full numerical solution to the problem is presented.

2. GURSON COMPRESSIBLE PLASTICITY MODEL

In this analysis we consider a rigid-perfectly plastic, i.e. non-hardening matrix material. Hutchinson (1987) provides a general treatment of Gurson's model applied to hardening materials. Based on the assumption that the matrix yields according to the von Mises yield condition, Gurson (1977) has developed a yield function Φ for a material which contains a dilute concentration of spherical voids:

$$\Phi = \frac{3\Sigma'_{ij}\Sigma'_{ij}}{2\sigma_y^2} + 2f\cosh\left(\frac{\Sigma_{kk}}{2\sigma_y}\right) - (1+f^2) = 0, \tag{1}$$

where σ_y is the current flow stress of the matrix material, Σ'_{ij} is the deviatoric stress acting on the voided material, $\Sigma_{kk}/3$ is the mean stress and f is the void concentration. Based on the assumption that the voids remain spherical during densification, the associated flow rule is given by

$$\dot{\varepsilon}_{ij} = \Lambda\frac{\partial\Phi}{\partial\Sigma_{ij}} = \frac{\Lambda}{\sigma_y}\left\{\frac{3\Sigma'_{ij}}{\sigma_y} + f\sinh\left(\frac{\Sigma_{kk}}{2\sigma_y}\right)\delta_{ij}\right\}, \tag{2}$$

where $\dot{\varepsilon}_{ij}$ is the strain rate tensor, Λ is a constant of proportionality and δ_{ij} is the Kronecker delta. In order to satisfy mass conservation, the porosity must evolve according to the relation

$$\dot{f} = (1-f)\dot{\varepsilon}_{kk}. \tag{3}$$

This formulation allows for a general hardening material in which σ_y would also be a function of the strain history. However, here we consider a perfectly plastic matrix material where σ_y is a constant and we introduce the following nondimensionalization:

$$\lambda \equiv \Lambda/\sigma_y \quad \sigma_{ij} \equiv \Sigma_{ij}/\sigma_y \tag{4}$$

so that eqns (1) and (2) become

$$\Phi = \frac{3\sigma'_{ij}\sigma'_{ij}}{2} + 2f\cosh\left(\frac{\sigma_{kk}}{2}\right) - (1+f^2) = 0 \tag{5}$$

and

$$\dot{\varepsilon}_{ij} = \lambda\frac{\partial\Phi}{\partial\sigma_{ij}} = \lambda\left\{3\sigma'_{ij} + f\sinh\left(\frac{\sigma_{kk}}{2}\right)\delta_{ij}\right\}. \tag{6}$$

3. DENSIFICATION UNDER SIMPLE MODES OF DEFORMATION

Some insight into the behavior of the porous material during densification can be gained by analysing a number of elementary deformation modes. In the limiting case of zero inclusion volume fraction, the deformation of the powder compact is purely hydrostatic. On the other hand, at the boundary of a spherical inclusion, the deformation is uniaxial strain since the radial displacement must be zero. Thus, in the limit of large inclusion volume fraction ($v_I \rightarrow 1$), the deformation is uniaxial throughout the matrix. To analyse these two cases, we consider an initially homogeneous body under uniform applied tractions so that the stress state is constant throughout the body. From eqn (6) it is clear that the strain history, and thus the porosity, are independent of position as well.

Uniform hydrostatic pressure

For a material without inclusions under applied pressure P, the stresses throughout the body will be hydrostatic, given by $\sigma_{ij} = -P\delta_{ij}$. The deviatoric stress vanishes and, from eqn (6), the strain increment is also hydrostatic, $\dot{\varepsilon}_{ij} = \dot{\varepsilon}\delta_{ij}$.

The governing equations for this deformation are

$$2f\cosh\left(\tfrac{3}{2}P\right) - (1+f^2) = 0 \tag{7}$$

$$\dot{\varepsilon} = -\lambda f \sinh\left(\tfrac{3}{2}P\right) \tag{8}$$

and

$$\dot{f} = 3(1-f)\dot{\varepsilon}. \tag{9}$$

Now eqn (7) can be solved for $P(f)$ and $f(P)$ explicitly:

$$P = \tfrac{2}{3}\operatorname{arccosh}\left\{\frac{1+f^2}{2f}\right\} = \tfrac{2}{3}\ln(1/f) \tag{10a}$$

or

$$f = \exp\left(-\tfrac{3}{2}P\right). \tag{10b}$$

From eqn (9) the relationship between an incremental porosity change and the corresponding incremental volume change is

$$\frac{\mathrm{d}V}{V} = \frac{\mathrm{d}f}{1-f}. \tag{11}$$

The energy per unit final (fully dense) volume required to densify from porosity f_0 to f under monotonically increasing hydrostatic pressure is calculated using eqns (10), (11) and $V_{\mathrm{m}} = V(1-f)$:

$$W(f_0, f) = -\frac{1}{V_{\mathrm{m}}}\int_{f_0}^{f} P(f)\,\mathrm{d}V = \frac{2}{3}\int_{f_0}^{f}\frac{\ln f}{(1-f)^2}\,\mathrm{d}f \tag{12}$$

which can be integrated to give

$$W(f_0, f) = \tfrac{2}{3}\ln\left(\frac{\psi(f)}{\psi(f_0)}\right), \tag{13a}$$

where

$$\psi(f) = (1-f)f^{f/(1-f)}. \tag{13b}$$

Note that as in the limit as $f \to 0$, $\psi(f) \to 1$, so that, while the pressure required to fully densify the material grows without bound [eqn (10)], the energy required to fully densify the material under hydrostatic loading is finite:

$$W(f_0, 0) = -\frac{2}{3}\left\{\ln(1-f_0) + \frac{f_0}{1-f_0}\ln f_0\right\}. \tag{14}$$

Equations (10) and (14), corresponding to uniform hydrostatic densification, are used to normalize the results obtained for more complex states of strain.

Uniaxial straining

Next consider the state $\dot{\varepsilon}_{11} \neq 0$ (all other strain increments are zero), which approximates the deformation of the matrix around an inclusion in the limit $v_{\rm I} \to 1$. Let $P = -\sigma_{11}$ and $S = -\sigma_{22}$ and note that, due to axial symmetry, $S = -\sigma_{33}$. Clearly then $\sigma_{\rm e}^2 = 3/2\sigma'_{ij}\sigma'_{ij} = (S-P)^2$ and $S' = 1/3(S-P)$. The yield condition (5) is thus

$$\sigma_{\rm e}^2 + 2f\cosh\left(\frac{\sigma_{kk}}{2}\right) - (1+f^2) = 0. \tag{15}$$

Setting $\dot{\varepsilon}_{22} = \dot{\varepsilon}_{33} = 0$ in eqn (6), we obtain

$$\sigma_{\rm e} = f\sinh\left(\frac{\sigma_{kk}}{2}\right). \tag{16}$$

Combining this result with eqn (15), we obtain an expression in the single unknown σ_{kk},

$$f^2\sinh^2\left(\frac{\sigma_{kk}}{2}\right) + 2f\cosh\left(\frac{\sigma_{kk}}{2}\right) - (1+f^2) = 0. \tag{17}$$

Using identities for the hyperbolic functions, this can be solved for $f(\sigma_{kk})$ as well as $\sigma_{kk}(f)$:

$$f = \frac{\cosh\left(\frac{\sigma_{kk}}{2}\right) + \sqrt{2}\sinh\left(\frac{\sigma_{kk}}{2}\right)}{1 - \sinh^2\left(\frac{\sigma_{kk}}{2}\right)} \tag{18}$$

$$\cosh\left(\frac{\sigma_{kk}}{2}\right) = \frac{-1+\sqrt{2+2f^2}}{f}. \tag{19}$$

From eqn (15) and (19) the normalized effective stress is

$$\sigma_{\rm e}^2 = 3 + f^2 - 2\sqrt{2+2f^2} \tag{20}$$

so that

$$P = \frac{2}{3}\left\{\operatorname{arccosh}\left(\frac{-1+\sqrt{2+2f^2}}{f}\right) + \sqrt{3+f^2-2\sqrt{2+2f^2}}\right\}. \tag{21}$$

The work done by a monotonically increasing pressure P is determined explicitly as

$$W(f_0, f) = -\int_{f_0}^{f} \frac{P(f)}{(1-f)^2}\,{\rm d}f, \tag{22}$$

where $P(f)$ is given by eqn (21).

4. DENSIFICATION AROUND A SPHERICAL INCLUSION

Plastic deformation during the densification of a powder compact which contains hard particles is modeled by considering a single spherical particle of radius a surrounded by a shell of dilutely voided matrix material of initial outer radius B. The initial inclusion volume fraction is thus $v_{\rm I0} = (a/B)^3$. The matrix is assumed to obey the Gurson yield condition (5) with the associated flow rule. It is assumed that the inclusions are much larger than the voids, yet they are small in comparison to the inter-particle spacing.

This spherical shell is deformed by application of a monotonically increasing radial stress $P = -\sigma_{rr}$ to the outside of the sphere which has a radius b at some instant during the deformation and thus has an instantaneous inclusion volume fraction $v_{\mathrm{I}} = (a/b)^3$.

The average porosity in the shell is

$$\bar{f} = \frac{V_{\mathrm{v}}}{V_{\mathrm{s}}}, \tag{23}$$

where V_{s} is the instantaneous volume of the shell and V_{v} is the current volume of voids in the shell. Alternatively, $\bar{f}$ may be determined by averaging the radial porosity distribution as

$$\bar{f} = \frac{1}{V_{\mathrm{s}}}\int_{V_{\mathrm{s}}} f\,\mathrm{d}V = \frac{3}{(b^3-a^3)}\int_a^b f(r)r^2\,\mathrm{d}r. \tag{24}$$

Finally, $\bar{f}$ can be determined as a function of the current shell volume in terms of the initial shell volume V_0 and initial average porosity $\bar{f}_0$:

$$\bar{f} = 1-\frac{V_0}{V_{\mathrm{s}}}(1-\bar{f}_0) = 1-\frac{B^3-a^3}{b^3-a^3}(1-\bar{f}_0). \tag{25}$$

This follows immediately from the incompressibility of the matrix material and inclusions. The relationship between initial and instantaneous inclusion volume fractions is thus

$$v_{\mathrm{I}} = \frac{v_{\mathrm{I0}}(1-\bar{f})}{1-\bar{f}_0+v_{\mathrm{I0}}(\bar{f}_0-\bar{f})}. \tag{26}$$

The fully dense inclusion volume fraction v_{If} is determined from eqn (26) with $\bar{f} = 0$ and the instantaneous volume fraction is given in terms of the final volume fraction by

$$v_{\mathrm{I}} = 1-\frac{1-v_{\mathrm{If}}}{1-\bar{f}v_{\mathrm{If}}}. \tag{27}$$

For the spherically symmetric problem the only non-zero stresses are σ_{rr} and $\sigma_{\theta\theta} = \sigma_{\phi\phi}$. It is useful to note that in the Gurson yield function and associated flow rule, the mean stress appears only in terms which are multiplied by f, while the von Mises effective stress only appears in porosity-independent terms. For this reason we can consider these stresses to be the fundamental stress unknowns for the problem, and thus make the change of variables:

$$D = \sigma_{rr}-\sigma_{\theta\theta}; \quad M = \tfrac{1}{2}\sigma_{kk}; \quad \sigma_{rr} = \tfrac{2}{3}(M+D); \quad \sigma_{\theta\theta} = \tfrac{1}{3}(2M-D). \tag{28}$$

The yield function (5) is thus

$$\Phi = D^2+2f\cosh M-(1+f^2) = 0. \tag{29}$$

The non-zero strain increments are radial and tangential, given by eqn (6):

$$\dot{\varepsilon}_{rr} = \lambda(2D+f\sinh M) \equiv \lambda f_r \tag{30}$$

$$\dot{\varepsilon}_{\theta\theta} = \lambda(-D+f\sinh M) \equiv \lambda f_\theta, \tag{31}$$

where f_r, f_θ are functions of stress introduced to simplify the notation. The porosity is incremented according to eqn (3):

$$\frac{\dot{f}}{1-f} = \dot{\varepsilon}_{kk} = \dot{\varepsilon}_{rr} + 2\dot{\varepsilon}_{\theta\theta}. \tag{32}$$

For the spherically symmetric stress state the equilibrium equation is

$$\frac{r}{2}\frac{\partial \sigma_{rr}}{\partial r} + (\sigma_{rr} - \sigma_{\theta\theta}) = 0, \tag{33}$$

where r is the Eulerian radial coordinate and σ_{ij} is the Cauchy stress. In terms of M and D, equilibrium is thus

$$\frac{r}{3}\frac{\partial(M+D)}{\partial r} + D = 0. \tag{34}$$

Compatibility for the spherically symmetric problem is expressed as $\dot{\varepsilon}_{rr} = \partial(r\dot{\varepsilon}_{\theta\theta})/\partial r$ or

$$rf_\theta \frac{\partial \lambda}{\partial r} + r\lambda \frac{\partial f_\theta}{\partial r} + \lambda(f_\theta - f_r) = 0. \tag{35}$$

It is possible to solve the compatibility equation for λ as a closed form function of stress and porosity by first noting that eqn (35) is a first-order separable equation in λ and then integrating:

$$\frac{\lambda}{\lambda_a} = \exp\left(- \int_a^r \frac{r\dfrac{\partial f_\theta}{\partial r} + (f_\theta - f_r)}{rf_\theta} \, \mathrm{d}r \right), \tag{36}$$

where λ_a is given by specifying the densification rate at the boundary through eqns (30)–(32) which is equivalent to specifying the applied stress. Thus, if at some instant we know the stress state, we can explicitly determine the strain increments in terms of the factor λ_a.

The rigid inclusion boundary condition is expressed as $u = v = \dot{\varepsilon}_{\theta\theta} = 0$ at $R = a$. This boundary condition is a state of uniaxial strain, thus we obtain expressions for M and D at $r = a$ as functions of the porosity at a from eqns (19) and (20):

$$D_a = -\sqrt{3 + f_a^2 - 2\sqrt{2 + 2f_a^2}}$$
$$M_a = -\operatorname{arccosh}\left\{ \frac{-1 + \sqrt{2 + 2f_a^2}}{f_a} \right\}. \tag{37}$$

Note that D_a is always negative and reaches its greatest magnitude at $f = 0$ where $D_a = 1 - \sqrt{2}$.

It is often mathematically convenient to consider f_a to be the independent variable rather than the applied external pressure P. Clearly under monotonically increasing P, the porosity at the boundary will be monotonically decreasing, thus there will be a one-to-one correspondence between P and f_a, and the two points of view are equivalent.

5. APPROXIMATE SOLUTIONS

The complete solution to this system of equations is to be solved by specifying some initial porosity distribution and then integrating to obtain porosity and stress as functions of both the radial coordinate r and the applied external pressure P. This must be done numerically. However, some insight into the behavior of the system can be gained by considering a number of simplifying cases.

We first consider the case of uniform porosity, that is $f \neq f(r)$. This may be used as an initial condition for the problem. Additionally, $f = 0$ is considered as a possible full density limiting case; however it will be seen that for constant $f \ll 1$, $\dot{f}$ is not independent of r. Thus, for any finite initial f, the solution will diverge from the constant f solution.

We next consider a uniform mean stress. For the large stresses encountered near full density, the deviation in the mean stress is small by comparison with the applied pressure. However, the constant mean stress solution is strictly valid only for a particular (non-uniform) porosity distribution.

Finally, we consider the full system of equations in the limit $f \to 0$. A solution is developed in the form $f \propto f_r(r)$ where $f_r(r)$ is independent of the applied pressure. This solution shows good agreement with the full numerical results.

In each approximate solution we determine D, M and f as functions of r and the porosity at the inclusion boundary f_a. Given these functions we then determine the applied pressure P as a function of the average porosity $\bar{f}$ for a given inclusion volume fraction. From the inclusion volume fraction we can determine the current outer radius b from eqn (25). f_a can then be determined for the specified $\bar{f}$ and b using eqn (24). Finally we calculate $P(b)$ from $D(b)$ and $M(b)$ using eqn (28). Note that the inclusion fraction may be specified as initial or final, as in either case we can determine the instantaneous volume fraction using eqn (26).

Uniform porosity

Typically we consider an initial undeformed state where the porosity is uniform. For this case the yield function can be differentiated to obtain $\partial M/\partial r$ as a function of D, its r derivative and the constant f:

$$\frac{\partial M}{\partial r} = \frac{2D\dfrac{\partial D}{\partial r}}{\sqrt{(1+f^2-D^2)^2-4f^2}}. \tag{38}$$

Substitution of this expression into equilibrium (34) yields a first-order separable o.d.e. in D and r (with f as a parameter)

$$\frac{r}{3}\left(1+\frac{2D}{\sqrt{(1+f^2-D^2)^2-4f^2}}\right)\frac{\partial D}{\partial r}+D = 0, \tag{39}$$

which can be integrated to obtain the closed form expression for r as a function of D:

$$\left(\frac{a}{r}\right)^3 = \frac{D}{D_a}\exp\left(2\int_{D_a}^{D} \mathrm{d}D/\sqrt{(1+f^2-D^2)^2-4f^2}\right). \tag{40}$$

Now M can be determined from eqn (29), thus we have determined M and D as implicit functions of r. It is notable that for $f \ll 1$, eqn (40) simplifies to a quadratic which can be solved explicitly for $D(r)$:

$$\left(\frac{a}{r}\right)^3 = \frac{D}{D_a}\frac{(1-D_a)}{(1+D_a)}\frac{(1+D)}{(1-D)}. \tag{41}$$

M is given by eqn (29) so that P is determined explicitly as

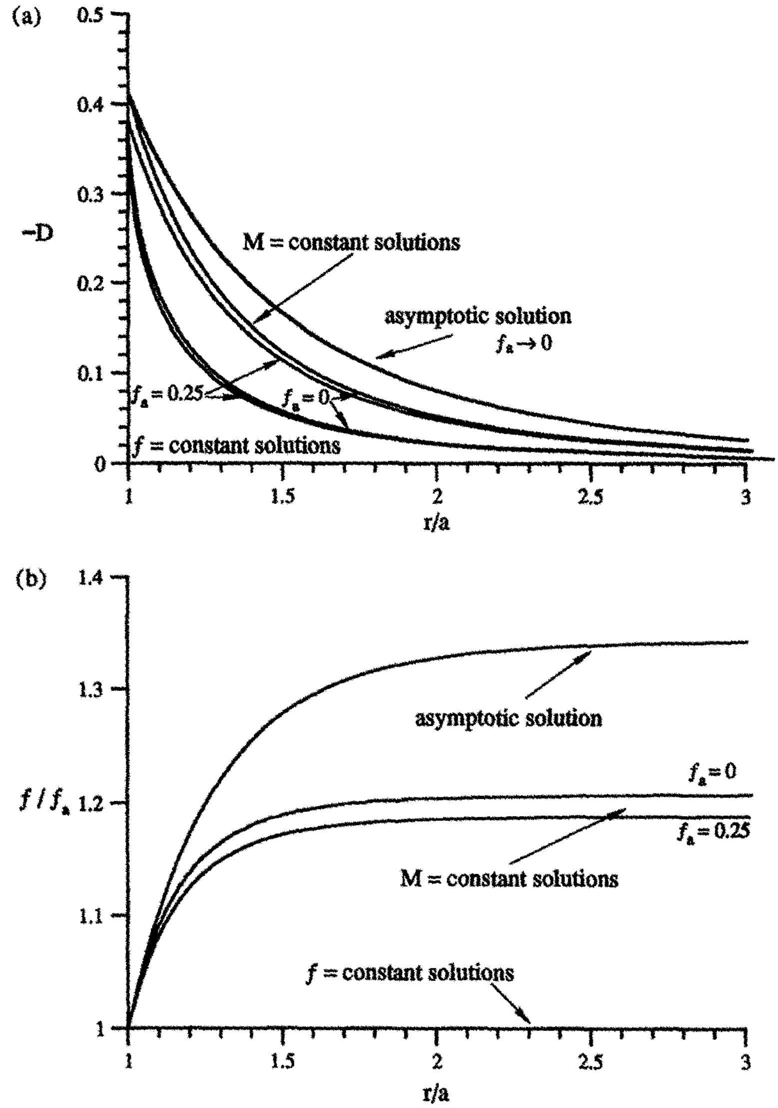

Fig. 1. Comparison of the deviatoric stress and porosity distributions for the three approximate solutions.

$$P = \frac{2}{3}\left\{\operatorname{arccosh}\left(\frac{1+f^2-D_b^2}{2f}\right)-D_b\right\}, \tag{42}$$

where D_b is a function of f and v_I given by eqn (40) or (41). Plots of D as a function of r obtained from eqns (40) and (41) for $f = 0.25$ and in the limit $f \to 0$, respectively, are shown in Fig. 1.

Uniform hydrostatic stress

Consider a mean stress M that is uniform in r, that is a function of the applied pressure P only. With this condition the equilibrium eqn (34) becomes simply

$$\frac{r}{3}\frac{\partial D}{\partial r}+D = 0, \tag{43}$$

which has the solution

$$\left(\frac{a}{r}\right)^3 = \frac{D}{D_a}. \quad (44)$$

Now if we specify f_a, we immediately obtain M_a and D_a from eqn (37), thus we know the full stress state as a function of r and f_a so that we can determine P as a function of v_I:

$$P = \frac{2}{3}\left\{\operatorname{arccosh}\left(\frac{-1+\sqrt{2+2f_a^2}}{f_a}\right) - v_\mathrm{I} D_a\right\}. \quad (45)$$

Finally, f is obtained as a function of r and f_a through eqn (29):

$$f = \cosh M - \sqrt{\cosh^2 M + D^2 - 1}. \quad (46)$$

$\bar{f}$ is then determined as an explicit function of f_a by evaluating eqn (47)

$$\bar{f} = \cosh M_a + \frac{v_\mathrm{I}}{1-v_\mathrm{I}}\left(\xi_1 - \xi_2 + D_a \ln\left\{v_\mathrm{I}\frac{\xi_2}{\xi_1}\right\}\right), \quad (47)$$

where

$$\xi_1 = D_a + \sqrt{\cosh^2 M_a - 1 + D_a^2}, \quad \xi_2 = D_a + \sqrt{(\cosh^2 M_a - 1)/v_\mathrm{I}^2 + D_a^2}. \quad (48)$$

Thus eqns (45) and (47) determine P as an implicit function of $\bar{f}$. Plots of D and f as a function of r obtained for the M = constant case from eqns (44) and (46) for $f_a = 0.25$ and in the limit $f_a \to 0$, are shown in Fig. 1.

Asymptotic solution for $f \to 0$

Differentiating the yield function (29) with respect to r, considering f, M and D all to be functions of r, and making the assumption that $f^2 \ll 1$ in eqns (A1) and (29), we arrive at a differential expression for f in terms of D and r

$$\frac{\partial f}{f} = -\left[3D + r\left\{1 - \frac{2D}{D^2-1}\right\}\frac{\partial D}{\partial r}\right]\frac{\partial r}{r}. \quad (49)$$

Under the assumption of infinitesimal f, the compatibility eqn (36) also simplifies to yield λ as a function of D and r

$$\frac{\partial \lambda}{\lambda} = 2\,\frac{3D + r(1-D)\dfrac{\partial D}{\partial r}}{D^2 - 1 - 2D}\,\frac{\partial r}{r}. \quad (50)$$

By letting $(1-f) \to 1$ in the porosity evolution eqn (32), differentiating with respect to r and combining with eqn (50), we obtain

$$\frac{\partial \dot{f}}{\dot{f}} = 2\,\frac{3D + r(1-D)\dfrac{\partial D}{\partial r}}{D^2 - 1 - 2D}\,\frac{\partial r}{r} + \frac{2D}{D^2-1}\,\frac{\partial D}{\partial r}\,\partial r. \quad (51)$$

Along with the stress eqn (49) we have developed a pair of coupled partial differential equations in D, f, $\dot{f}$ and r.

Now if we look for solutions in which f is separable into the product of functions of r and P as $f = f_a f_r(r)$, then clearly $\dot{f} = f_r(r)\dot{f}_a$ and we can write

$$\frac{\partial \dot{f}}{\dot{f}} = \frac{\partial f}{f}. \tag{52}$$

By equating expressions (51) and (52), we obtain a single ordinary differential equation for $D(r)$ which can be integrated to obtain a closed form expression for $r(D)$:

$$\left(\frac{a}{r}\right)^3 = \frac{D}{D_a}\exp\left(\frac{2(D_a-D)}{(D_a-1)(D-1)}\right), \tag{53}$$

where $D_a \to 1-\sqrt{2}$ as $f \to 0$ from eqn (37). Additionally, we obtain f as a function of D, and therefore r, through eqn (53):

$$\frac{f}{f_a} = \left(\frac{1+D}{1+D_a}\right)\left(\frac{1-D_a}{1-D}\right)\exp\left\{\frac{2(D_a-D)}{(D_a-1)(D-1)}\right\}, \tag{54}$$

where f_a is the (pressure-dependent) porosity at the rigid inclusion boundary. Now to complete the analysis we need to determine the porosity at $r = b$, f_b in terms of $\bar{f}$ so that we can calculate M from the yield condition. From eqn (54) we determine f as a function of $D(r)$ and f_b:

$$\frac{f}{f_b} = \left(\frac{1+D}{1+D_b}\right)\left(\frac{1-D_b}{1-D}\right)\exp\left\{\frac{2(D_b-D)}{(D_b-1)(D-1)}\right\}. \tag{55}$$

We then obtain the volume average porosity $\bar{f}$ from eqn (24),

$$\bar{f} = \frac{1}{b^3-a^3}\int_a^b f(r)r^3 3\frac{\mathrm{d}r}{r}, \tag{56}$$

where r^3 is a known function of D from eqn (53) and $3\mathrm{d}r/r$ is given by eqn (A15). Therefore, we obtain a closed form expression for $\bar{f}$:

$$\frac{\bar{f}}{f_b} = \frac{D_a-D_b}{1-v_1}\frac{1-D_b}{1+D_b}\left[\frac{1}{D_a}+\frac{2D_b(2D_b-D_aD_b+2D_a-3)}{(D_a-1)^2(D_b-1)^2}\right]. \tag{57}$$

It is important to recognize that this is a function of the inclusion volume fraction only since D_b is known from eqn (53). Now from the yield condition we can derive P as a function of D_b and f_b and thus as a function of the inclusion volume fraction and the average porosity:

$$P = \frac{2}{3}\left\{\operatorname{arccosh}\left(\frac{1-D_b^2}{2f_b}\right)-D_b\right\}. \tag{58}$$

The deviatoric stress porosity distributions for the asymptotic solution are shown in Fig. 1 for comparison with the uniform mean stress and uniform porosity solutions. The asymptotic solution shows the greatest influence of the inclusion at any volume fraction. Thus, use of the simpler approximations could lead to under-estimating the pressure required for densification.

Comparison of solutions for small f

Figure 2 shows a comparison of P/P_h for the three analytic solutions as well as the results of the numerical calculation. The pressure vs porosity relations derived for the three approximate solutions are each of the form

$$P = \frac{2}{3}\left\{\operatorname{arccosh}\left(\frac{\beta(v_I, \bar{f})}{2\bar{f}}\right) - D_b(v_I, \bar{f})\right\}, \tag{59}$$

where β and D_b are weak functions of $\bar{f}$. For small $\bar{f}$, the inverse hyperbolic can be approximated by a natural log

$$P \cong \tfrac{2}{3}\{\ln(\beta/\bar{f}) - D_b\} \tag{60}$$

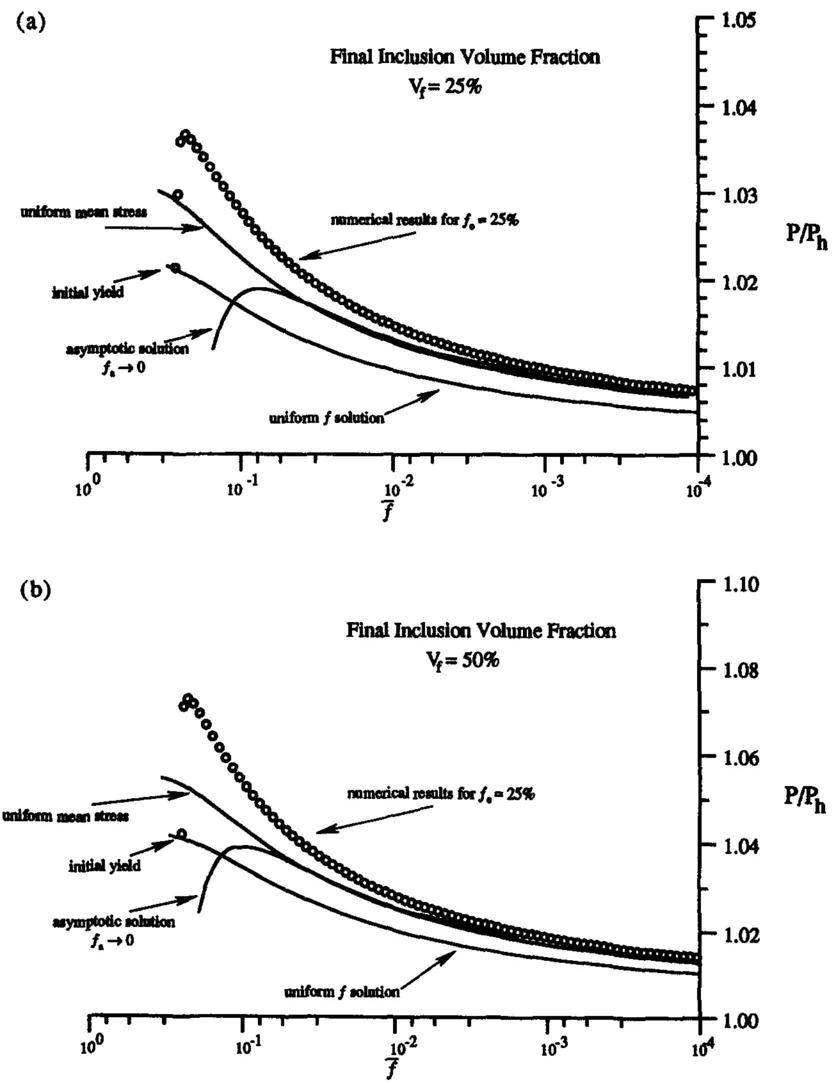

Fig. 2. Applied pressure as a function of average porosity for 25% and 50% final, or fully dense, inclusion volume fractions. Uniform porosity, uniform stress and asymptotic solutions are compared with the complete numerical solution with a uniform 25% initial porosity.

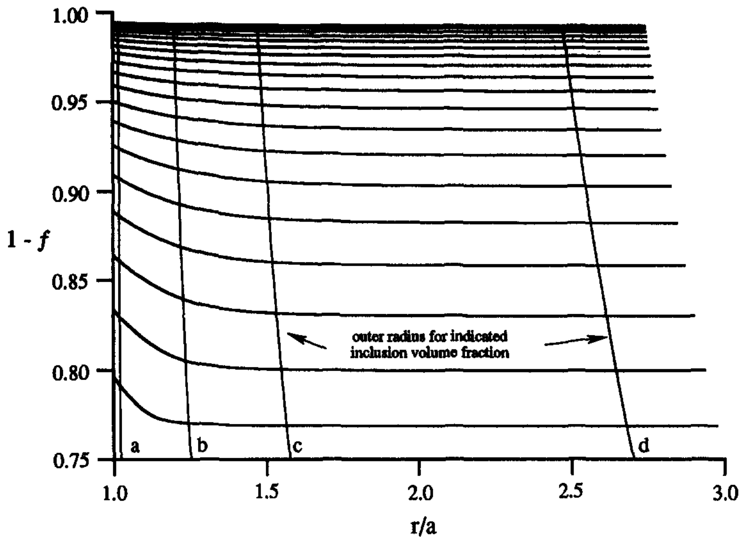

Fig. 3. Radial porosity distributions for an initial matrix porosity $f_0 = 0.25$. Curves represent approximately equal applied pressure increments. Lines a, b, c, d show the path of the outer boundary for the initial inclusion fractions $v_{I0} = 0.9, 0.5, 0.25, 0.05$, respectively. These volume fractions are also referenced in Figs 4–5.

so that we can normalize by the exact expression for $v_I = 0$ [eqns 10a, b]:

$$\frac{P}{P_h} = 1 + \frac{\ln \beta - D_b}{\ln(1/\bar{f})} = 1 + \frac{\alpha}{\ln(1/\bar{f})}. \tag{61}$$

For the uniform porosity approximation, α is given by

$$\alpha = \ln(1 + f^2 - D_b^2) - D_b, \tag{62}$$

where D_b is found from eqn (40) or (41). It can clearly be seen from Fig. 1 that D is a weak function of f so that α depends primarily on the inclusion volume.

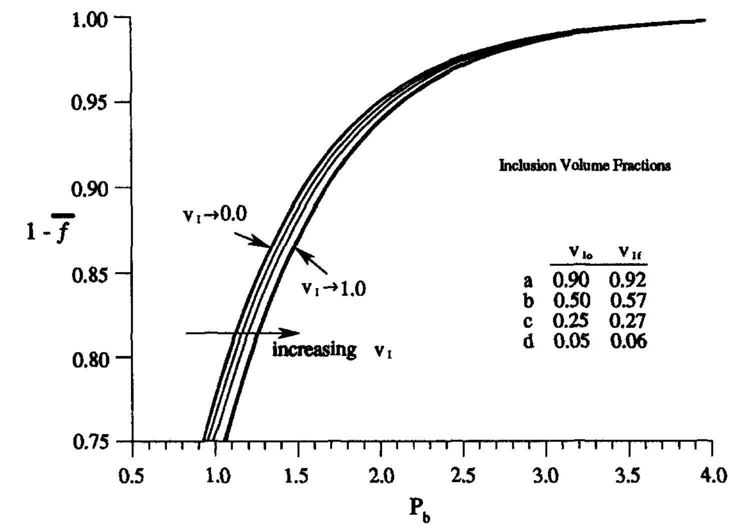

Fig. 4. Densification with initial matrix porosity $f_0 = 0.25$ for various initial inclusion volume fractions. Curves for $v_{I0} = 0$ and $v_{I0} = 1$ are obtained from eqns (10) and (21).

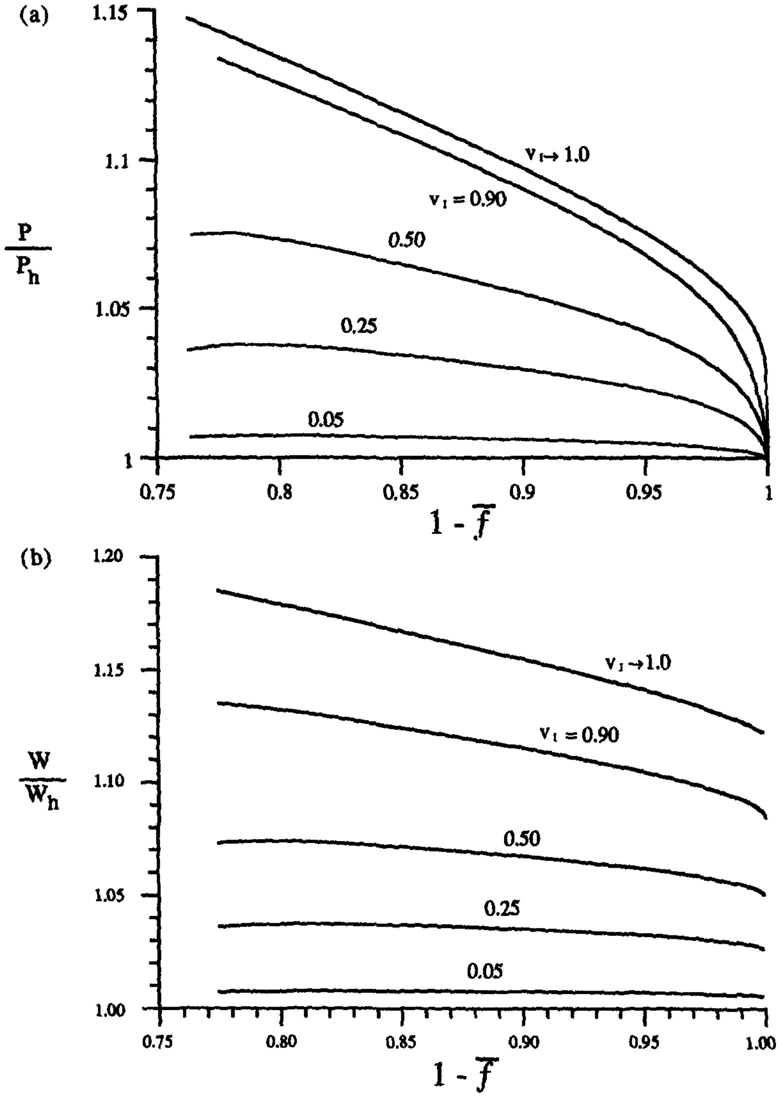

Fig. 5. Pressure and work to densify normalized by corresponding quantities for pure hydrostatic densification ($v_I = 0$) given by eqns (10) and (13).

For the uniform mean stress solution, α is given by

$$\alpha = \ln\left(2\frac{\sqrt{2+2f_a^2}-1}{f_a/\bar{f}}\right)-v_I D_a, \tag{63}$$

where f_a must be determined as a function of $\bar{f}$ through eqn (47). This is a very cumbersome formulation.

Finally, for the asymptotic solution, α is

$$\alpha = \ln\{\bar{f}/f_b(1-D_b^2)\}-D_b, \tag{64}$$

where $\bar{f}/f_b$ is given as an explicit function of D_b by eqn (57) and D_b is a function of inclusion volume fraction only through eqn (53).

6. NUMERICAL SOLUTIONS

A full solution to the system [eqns (29)–(37)] was obtained by an explicit integration technique. The system of equations is completed by the specification of an initial porosity

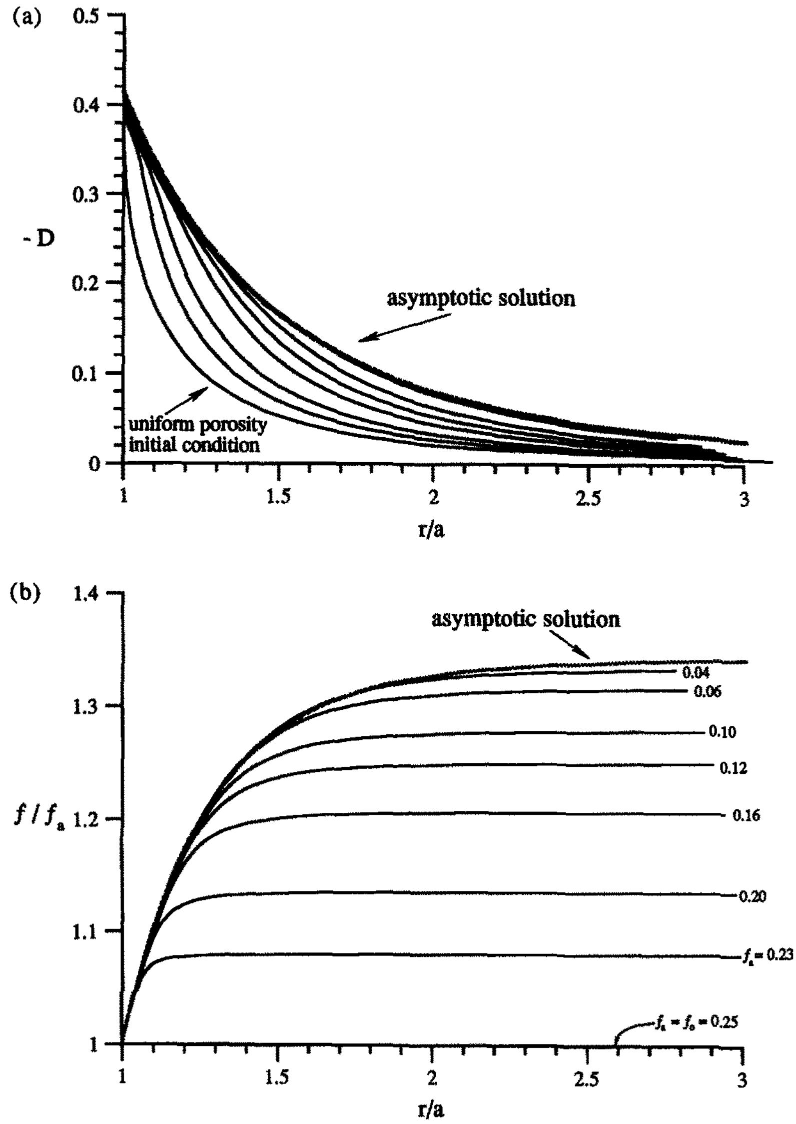

Fig. 6. Deviatoric stress and porosity distributions during densification for an initial matrix porosity $f_0 = 0.25$. Asymptotic curves are from eqns (53) and (54).

distribution, as well as an applied load history. Here we assume a uniform initial porosity and increment the porosity at the rigid boundary from its initial value to zero.

Using the inner boundary state, as opposed to the more intuitive external applied pressure, as a load parameter has two distinct advantages. Most importantly this allows the radial integration at each load increment to be formulated as an explicit "marching scheme", whereas specifying the applied pressure would require an iterative technique to match the inner boundary condition. Moreover, the integration can be carried out to an arbitrary external radius, thus providing solutions for a continuous range of inclusion volume fractions from a single integration. The details of the numerical calculation are given in Appendix B.

Figure 3 shows a typical densification history for a 25% uniform initial porosity. The plot is constructed such that the increment in P at $r = 3$ is a constant. Initially the most rapid densification takes place at the inclusion boundary, however subsequent densification is more uniform. The lines marked as a, b, c and d represent the path of a material point at the outer boundary for a particular initial, or final, inclusion volume fraction. The key to this is shown inset in Fig. 4, which is constructed by determining $\bar{f}$ from eqn (25) for

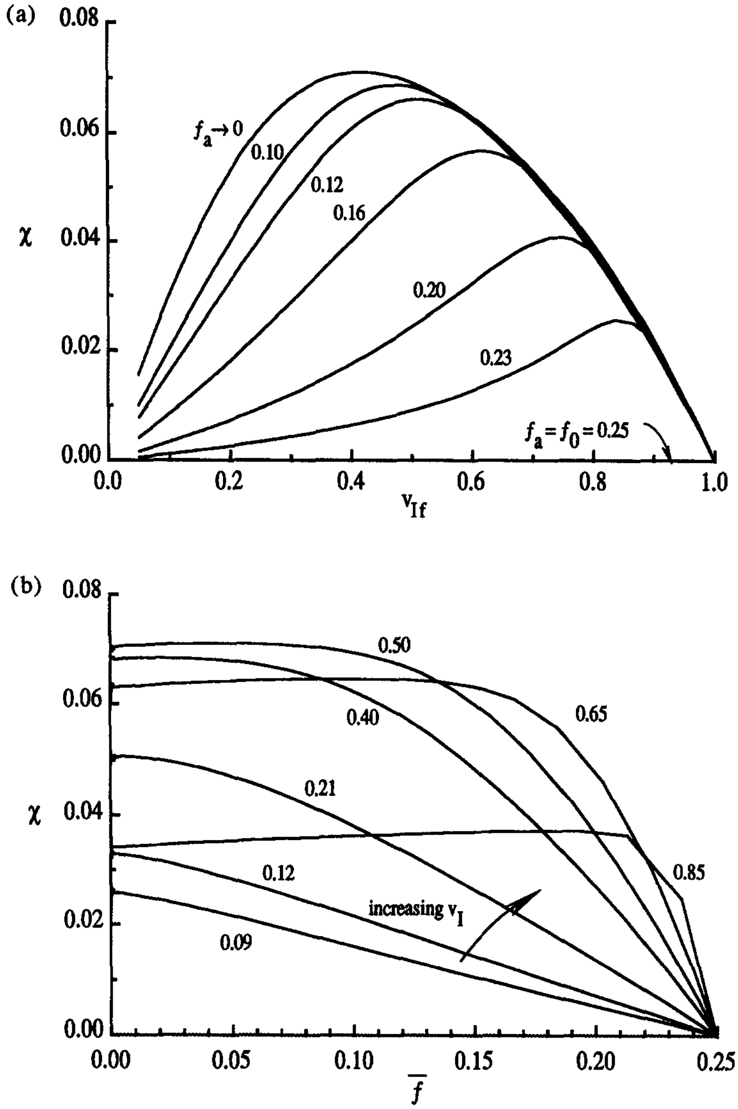

Fig. 7. Development of the porosity defect parameter χ during densification for an initial uniform marix porosity $f_0 = 0.25$. χ is given by eqn (65).

each of these points. It can be seen that the pressure required to yield the material at any given $\bar{f}$ is bounded by the pressure required to deform the material under hydrostatic and uniaxial strain conditions. As these bounds are fairly tight, the effect of inclusion volume fraction is small.

Figure 5(a) illustrates the volume fraction effect by normalizing the same data from Fig. 4 by the pure hydrostatic pressure required to yield the porous material at the same void volume fraction. The pressure required in the composite is at most 15% greater than the hydrostatic pressure which consolidates an unreinforced material. In the limit as full densification is approached all of the solutions converge. Note that we are normalizing by a quantity which is singular as $f \to 0$. Figure 5(b) shows the plastic work per unit fully dense matrix volume performed during the densification as determined by eqn (12) and normalized by the hydrostatic work from eqn (13). These quantities are all bounded as $f \to 0$ and we see that there is a finite increase in the work required to fully densify a quantity of material with increasing inclusion concentration.

Figure 6 shows the deviatoric stress and porosity distributions obtained from the numerical solution for a 25% initial porosity. The numerical results clearly approach the asymptotic solution, which is shown for comparison.

The effect of the non-uniform porosity distribution can be more clearly seen by defining a defect parameter in terms of the ratio of the maximum porosity to the average porosity:

$$\chi = 1 - f_{\max}/\bar{f}. \tag{65}$$

For this problem the largest porosity is always far from the inclusion. Figure 7 shows χ as a function of inclusion volume fraction and as a function of average porosity for a number of inclusion fractions. The defect parameter clearly diminishes as the volume fraction becomes small, however for a fraction as small as 20%, the defect will be of the order of 5%.

7. CONCLUSIONS

In terms of observable quantities, i.e. the applied pressure P_b and the average porosity $\bar{f}$, which equals one minus the average relative density, the implication of these results are that the pressures required during densification of a material containing rigid particulates will be at most 10% greater than those required to densify the same material without the inclusions. The extreme situation corresponds to densification under uniaxial straining. Likewise, the work required to densify a volume of matrix material is shown to be bounded by the work to densify the same material under uniaxial strain conditions, which is at most 20% greater than the work required to densify under hydrostatic straining. However, the porosity in the matrix in regions far from inclusions may be as much as 7% greater than the volume average.

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APPENDIX A: ASYMPTOTIC SOLUTION FOR $f \to 0$

Differentiating eqn (29) with respect to r and considering f, M and D all to be functions of r leads to the expression

$$\frac{\partial M}{\partial r} = \frac{2D\dfrac{\partial D}{\partial r} + (1-f^2-D^2)\dfrac{1}{f}\dfrac{\partial f}{\partial r}}{\sqrt{(1+f^2-D^2)^2-4f^2}}. \tag{A1}$$

This expression is generally valid for large f, however it has been assumed that the mean stress is compressive everywhere. We now make the assumption that $f^2 \ll 1$ in eqns (A1) and (29):

$$\frac{\partial M}{\partial r} = \frac{2D\dfrac{\partial D}{\partial r} + (1-D^2)\dfrac{1}{f}\dfrac{\partial f}{\partial r}}{1-D^2} \tag{A2}$$

$$2f\cosh M = 1-D^2. \tag{A3}$$

Substitution of eqn (A2) into the equilibrium eqn (34) yields

$$0 = \frac{2D\,\partial D}{(1-D^2)\,\partial r} + \frac{1}{f}\frac{\partial f}{\partial r} + \frac{\partial D}{\partial r} + \frac{3D}{r}. \tag{A4}$$

We can also use eqn (A3) to calculate the sinh M function for $f \ll 1$:

$$\sinh M = \pm\sqrt{\cosh^2 M - 1} = \pm\sqrt{\left[\frac{1-D^2}{2f}\right]^2 - 1} \cong \pm\left[\frac{1-D^2}{2f}\right] = -\cosh M, \tag{A5}$$

where we have used the assumption that the mean stress M is compressive, as well as the requirement that $D^2 \neq 1$. It can be argued that D reaches its largest (negative) value at the rigid boundary and asymptotically goes to zero with increasing r, hence it can be easily shown from the boundary condition that $1-\sqrt{2} < D < 0$. Now we can make the substitution

$$f \sinh M = \frac{D^2-1}{2} \tag{A6}$$

into the flow rule eqns (30) and (31):

$$\dot{\varepsilon}_{rr} = \lambda\left(2D + \frac{D^2-1}{2}\right) = \lambda f_r \tag{A7}$$

$$\dot{\varepsilon}_{\theta\theta} = \lambda\left(-D + \frac{D^2-1}{2}\right) = \lambda f_\theta. \tag{A8}$$

Significantly, we see that f_r and f_θ are asymptotically functions of D only, thus the ratio of the strain increments is determined solely by the deviatoric stress. If we put these expressions into the compatibility eqn (36) we see that $d\lambda/\lambda$ is now an explicit function of D and r as $f \to 0$:

$$\frac{\partial\lambda}{\lambda} = 2\frac{3D + r(1-D)\frac{\partial D}{\partial r}}{D^2-1-2D}\frac{\partial r}{r}. \tag{A9}$$

By letting $(1-f) \to 1$ in the porosity evolution eqn (32);

$$\dot{f} = \lambda(f_r + 2f_\theta) = \tfrac{3}{2}\lambda(D^2-1) \tag{A10}$$

and by differentiating with respect to r we obtain

$$\frac{\partial\dot{f}}{\dot{f}} = \frac{\partial\lambda}{\lambda} + \frac{2D}{D^2-1}\frac{\partial D}{\partial r}\partial r \tag{A11}$$

and finally, combining eqn (A11) with eqn (A9),

$$\frac{\partial\dot{f}}{\dot{f}} = 2\frac{3D + r(1-D)\frac{\partial D}{\partial r}}{D^2-1-2D}\frac{\partial r}{r} + \frac{2D}{D^2-1}\frac{\partial D}{\partial r}\partial r. \tag{A12}$$

Along with the stress eqn (A4) (solved for $\partial f/f$) we have developed a pair of coupled partial differential equations in D, f, $\dot{f}$ and r:

$$\frac{\partial f}{f} = -\left[3D + r\left\{1 - \frac{2D}{D^2-1}\right\}\frac{\partial D}{\partial r}\right]\frac{\partial r}{r}. \tag{A13}$$

Now if we look for solutions in which f is separable into the product of functions of r and P as $f = f(r)f_P(P)$ then clearly $\dot{f} = f(r)\dot{f}_P$ and we can write

$$\frac{\partial\dot{f}}{\dot{f}} = \frac{\partial f}{f}. \tag{A14}$$

Two implications of this are that D is a function of r only and that λ is also separable into the product of an r-dependent term and a pressure-dependent term. By equating expressions (A12) and (A13), we obtain a single ordinary differential equation for $D(r)$:

$$-\left[3D + r\left\{1 - \frac{2D}{D^2-1}\right\}\frac{\partial D}{\partial r}\right]\frac{\partial r}{r} = 2\frac{3D + r(1-D)\frac{\partial D}{\partial r}}{D^2-1-2D}\frac{\partial r}{r} + \frac{2D}{D^2-1}\frac{\partial D}{\partial r}\partial r$$

$$-\left[3D + r\{1\}\frac{\partial D}{\partial r}\right]\frac{\partial r}{r} = 2\frac{3D + r(1-D)\frac{\partial D}{\partial r}}{D^2-1-2D}\frac{\partial r}{r}$$

$$-\left[3D+r\frac{\partial D}{\partial r}\right][D^2-1-2D] = 6D+2r(1-D)\frac{\partial D}{\partial r}$$

$$3\frac{\partial r}{r} = \frac{-D^2-1+4D}{D(D-1)^2}\partial D. \tag{A15}$$

Integration of eqn (A15) results in a closed form expression for $r(D)$,

$$\left(\frac{a}{r}\right)^3 = \frac{D}{D_a}\exp\left(\frac{2(D_a-D)}{(D_a-1)(D-1)}\right), \tag{A16}$$

where $D_a \to 1-\sqrt{2}$ as $f \to 0$ from eqn (37). Additionally, eqns (A15) and (A13) can be combined to obtain

$$\frac{\partial f}{f} = \left[-\frac{2D}{(D-1)^2}+\frac{2D}{D^2-1}\right]\partial D, \tag{A17}$$

which is easily integrated to yield f as a function of D, and therefore r, through eqn (A16):

$$\frac{f}{f_a} = \left(\frac{1+D}{1+D_a}\right)\left(\frac{1-D_a}{1-D}\right)\exp\left(\frac{2(D_a-D)}{(D_a-1)(D-1)}\right), \tag{A18}$$

where f_a is the (pressure-dependent) porosity at the rigid inclusion boundary. Of course, given f and D, M and thus all of the stress components are found as functions of r through eqn (A3).

APPENDIX B: NUMERICAL SCHEME

The integration is carried out by discretizing the $r-f_b$ space and writing the equations such that X_{ij}, representing the unknown value of the variable X, depends only on the previously calculated values X_{i-1j} and X_{ij-1}. We have used a forward Euler integration in the radial coordinate so that

$$\left.\frac{\partial X}{\partial r}\right|_{i-1j} = \frac{X_{ij}-X_{i-1j}}{r_{ij}-r_{i-1j}}, \tag{B1}$$

where r_{ij} is the current Eulerian coordinate of point ij. Incremental integration was performed using backwards Euler derivatives, as this proved to be most stable

$$\dot{X}_{ij} = X_{ij}-X_{ij-1}. \tag{B2}$$

The formulation utilized contains only first derivatives.

With this discretization of the derivatives we obtain a non-linear system of equations in the primary unknowns σ_{rr}, $\sigma_{\theta\theta}$ and u at point ij. f, $\dot{\varepsilon}_{rr}$, $\dot{\varepsilon}_{\theta\theta}$, f_r, f_θ and r are all explicit functions of the primary unknowns. Because we have used a forward difference in r, the (linear) expressions for u, r and σ_{rr} decouple from the remainder of the system:

$$u_i = u_{i-1}+(R_i-R_{i-1})\left\{\frac{1-f_0}{1-f_{i-1}}\frac{(R_{i-1}+u_{i-1})^2}{R_{i-1}u_{i-1}}-1\right\} \tag{B3}$$

$$r_i = R_i+u_i \tag{B4}$$

$$\sigma_{rri} = \sigma_{rri-1}+2\left[\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}\right]_{i-1}(r_i-r_{i-1}), \tag{B5}$$

where R is the undeformed coordinate, f_0 is the initial porosity, and all variables are evaluated at the current increment so that the second subscript can be omitted without ambiguity. In deriving these expressions we have integrated eqn (32) to obtain

$$-\ln\left(\frac{1-f}{1-f_0}\right) = \varepsilon_{kk} = \varepsilon_{rr}+2\varepsilon_{\theta\theta} \tag{B6}$$

and used the finite deformation expressions for ε in terms of the Eulerian coordinate r,

$$\varepsilon_{rr} = -\ln(1-\partial u/\partial r) \tag{B7}$$

$$\varepsilon_{\theta\theta} = -\ln(1-u/r), \tag{B8}$$

as well as the equilibrium eqn (33).

The remaining unknowns at node ij can now be determined by specifying $\sigma_{\theta\theta}|_{ij}$. f is determined from eqn (46), f_r and f_θ form their definitions [eqns (30), (31)], $\varepsilon_{\theta\theta}$ is determined from eqn (B8) and finally ε_{rr} is found from eqn (B6). $\sigma_{\theta\theta}$ is then determined iteratively by satisfying the flow rule

$$\frac{\dot{\varepsilon}_{rr}}{\dot{\varepsilon}_{\theta\theta}} = \frac{f_r}{f_\theta}, \tag{B9}$$

where the backward Euler expression for the strain increments (B2) has been applied.